

On the string Lie algebra

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Abstract

We construct an abelian representative for the crossed module associated to the string Lie algebra. We show how to apply this construction in order to define quasi-invariant tensors which serve to categorify the infinitesimal braiding on the category of \mathfrak{g} -modules given by an r -matrix, following Cirio-Martins [2].

Introduction

Given a simple compact simply connected Lie group G , the string group $\text{Str}(G)$ is in some sense its 3-connected cover. There are various models for $\text{Str}(G)$ in the literature, for example as a weak 2-group with finite-dimensional components or as a strict 2-group with infinite-dimensional components [1]. But strict 2-groups are nothing else than crossed modules, thus the string group and its Lie algebra, the *string Lie algebra*, can be represented by crossed modules, see for example [7] for some model. Crossed modules of Lie algebras are closely related to 3-cohomology classes, and the 3-cohomology class corresponding to the string Lie algebra is the class represented by the *Cartan cocycle* $\kappa([-,-],) \in Z^3(\mathfrak{g}, \mathbb{C})$ for a simple complex Lie algebra \mathfrak{g} and its symmetric, non-degenerate bilinear Killing form κ . Introductory material on crossed modules is given in Section 1, and material on the string Lie algebra in Section 3 (closely following [7]).

We want to emphasize in this article that it is interesting to find *abelian representatives* for a given equivalence class of crossed modules. These are crossed modules of the form $\mu : V_2 \rightarrow V_3 \times_{\alpha} \mathfrak{g}$ which are spliced together from a short exact sequence of \mathfrak{g} -modules

$$0 \rightarrow V_1 \rightarrow V_2 \xrightarrow{d} V_3 \rightarrow 0$$

and an abelian extension $V_3 \times_{\alpha} \mathfrak{g}$ of \mathfrak{g} by V_3 via a 2-cocycle α . We will almost always consider crossed modules where $V_1 = \mathbb{C}$, the trivial \mathfrak{g} -module. The construction of abelian representatives stems back to [8] and is recalled here in Section 2.

Unfortunately, the explicit expression for the 2-cocycle α which the second author presented in [8] in order to construct an abelian representative for the Cartan cocycle corresponding to a general simple complex Lie algebra is wrong. We thank Lucio Cirio for this observation. In the present paper, we propose a new expression for α (see Equation 9) which is kind of abstract, but which works for any constant valued 3-cocycle. The formula relies on an explicit formula for the homotopy contracting the Koszul complex and appeared first (to our knowledge) in the first author's doctoral thesis. In order to render its expression explicit, one has to compute the Eulerian idempotent on the corresponding universal enveloping algebra, and this is in general a difficult problem. We illustrate this at the end of Section 4.

Cirio and Martins applied the abelian representative of [8] for $\mathfrak{sl}_2(\mathbb{C})$ which is given in terms of vector fields and formal differential forms to the problem of categorifying the infinitesimal braiding in the category of \mathfrak{g} -modules which is associated to an r-matrix for \mathfrak{g} , i.e. a symmetric \mathfrak{g} -invariant tensor $r \in \mathfrak{g} \otimes \mathfrak{g}$. They showed that *quasi-invariant tensors* (see Definition 5) lead to such a braiding. As an application of our theory, we show in Section 5 how to construct quasi-invariant tensors for a wide class of Lie algebras, including all simple complex Lie algebra, and therefore extending Cirio and Martins' construction.

For this, we had to formalize one compatibility condition which was automatically satisfied in Cirio-Martins' framework. Let $\mu : V_2 \rightarrow V_3 \times_{\alpha} \mathfrak{g}$ be an abelian representative for a crossed module corresponding to a trivial-valued 3-cocycle γ . Take a linear section Q of the map $d : V_2 \rightarrow V_3$, and lift α to $w := Q \circ \alpha$. Take its Chevalley-Eilenberg coboundary $d_{CE}w =: \Phi$. On the other hand, consider a \mathfrak{g} -invariant symmetric 2-tensor $r = \sum_i s_i \otimes t_i$. Then the condition reads for all $X, Y \in \mathfrak{g}$:

$$\sum_i \Phi(s_i, X, Y) \otimes t_i = 1_{V_2} \otimes [X, Y], \quad \text{and} \quad \sum_i s_i \otimes \Phi(t_i, X, Y) = [X, Y] \otimes 1_{V_2},$$

where 1_{V_2} generates the image of $V_1 = \mathbb{C}$ in V_2 .

Theorem 1. *Suppose that in the above situation the above compatibility condition is satisfied.*

Then the triple (\bar{r}, ξ, c) (defined in Section 5.2) is a quasi-invariant tensor for the crossed module $\mu : V_2 \rightarrow V_3 \times_{\alpha} \mathfrak{g}$.

We show large classes of examples where this condition is satisfied, in particular it is satisfied for all (finite-dimensional) simple complex Lie algebras.

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1 Crossed modules

Fix a field \mathbb{K} . Mainly we will consider $\mathbb{K} = \mathbb{C}$.

Definition 1. A crossed module of Lie algebras is a homomorphism of Lie algebras $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$ together with an action of \mathfrak{n} on \mathfrak{m} by derivations, denoted $m \mapsto n \cdot m$, such that

- (a) $\mu(n \cdot m) = [n, \mu(m)]$ for all $n \in \mathfrak{n}$ and all $m \in \mathfrak{m}$,
- (b) $\mu(m) \cdot m' = [m, m']$ for all $m, m' \in \mathfrak{m}$.

Remark 1. To each crossed module of Lie algebras $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$, one associates a four term exact sequence

$$0 \rightarrow V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \xrightarrow{\pi} \mathfrak{g} \rightarrow 0$$

where $\ker(\mu) =: V$ and $\mathfrak{g} := \text{coker}(\mu)$.

Remark 2. • By (a), \mathfrak{g} is a Lie algebra, because $\text{im}(\mu)$ is an ideal.

- By (b), V is a central Lie subalgebra of \mathfrak{m} , and in particular abelian.
- By (a), the action of \mathfrak{n} on \mathfrak{m} induces a structure of a \mathfrak{g} -module on V .
- Note that in general \mathfrak{m} and \mathfrak{n} are not \mathfrak{g} -modules.

Let us recall how to associate to a crossed module a 3-cocycle of \mathfrak{g} with values in V . Recall the exact sequence from Remark 1:

$$0 \rightarrow V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \xrightarrow{\pi} \mathfrak{g} \rightarrow 0$$

The first step is to take a linear section ρ of π and to compute the defect of ρ to be a Lie algebra homomorphism, i.e.

$$\alpha(x_1, x_2) := [\rho(x_1), \rho(x_2)] - \rho([x_1, x_2]).$$

Here, $x_1, x_2 \in \mathfrak{g}$. α is bilinear and skewsymmetric in x_1, x_2 . We have obviously $\pi(\alpha(x_1, x_2)) = 0$, because π is a Lie algebra homomorphism, so $\alpha(x_1, x_2) \in \text{im}(\mu) = \ker(\pi)$. By exactness, there exists $\beta(x_1, x_2) \in \mathfrak{m}$ such that

$$\mu(\beta(x_1, x_2)) = \alpha(x_1, x_2).$$

Choosing a linear section σ on $\text{im}(\mu)$, one can choose β as

$$\beta(x_1, x_2) = \sigma(\alpha(x_1, x_2))$$

showing that we can suppose β bilinear and skewsymmetric in x_1, x_2 .

Now, one computes $\mu(d^{\mathfrak{m}}\beta(x_1, x_2, x_3)) = 0$, where $d^{\mathfrak{m}}$ is the formal expression of the Lie algebra coboundary operator, while the map $\eta \circ \rho$ is not an action in general.

Choosing a linear section τ on $i(V) = \ker(\mu)$, one can choose γ to be

$$\gamma := \tau \circ d^m \beta \quad (1)$$

gaining that γ is also trilinear and skewsymmetric in x_1, x_2 and x_3 . One computes that γ is a 3-cocycle of \mathfrak{g} with values in V .

On the other hand, one may introduce an equivalence relation on the class of crossed modules and an abelian group structure. With this structure at hand, the above construction of the cocycle γ leads to the following theorem.

Theorem 2 (Gerstenhaber). *There is an isomorphism of abelian groups*

$$b : \text{crmod}(\mathfrak{g}, V) \cong H^3(\mathfrak{g}, V).$$

A proof of this theorem is (more or less) contained in [3] and [4]. According to [6], we attribute it to Murray Gerstenhaber. We refer to [8] for further computational details on this theorem.

The main question we want to address here is how to construct crossed modules which relate to a given 3-cocycle under the correspondence of this theorem.

2 Construction of crossed modules

2.1 Construction using a short exact sequence of modules

Let us ask the following question: Given a Lie algebra \mathfrak{g} , a short exact sequence of \mathfrak{g} -modules

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0 \quad (2)$$

(regarded as a short exact sequence of abelian Lie algebras) and an abelian extension \mathfrak{r} of \mathfrak{g} by the abelian Lie algebra V_3

$$0 \rightarrow V_3 \rightarrow \mathfrak{r} \rightarrow \mathfrak{g} \rightarrow 0, \quad (3)$$

is the Yoneda product of (2) and (3) a crossed module? In case (3) is given by a 2-cocycle α , we use the notation $\mathfrak{r} = V_3 \times_\alpha \mathfrak{g}$.

Theorem 3. *In the above situation, the Yoneda product of (2) and (3), namely*

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow \mathfrak{r} = V_3 \times_\alpha \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow 0,$$

is a crossed module, the associated 3-cocycle of which is (cohomologous to) the image $\partial\alpha$ of α , i.e. the 2-cocycle defining the central extension (3), under the connecting homomorphism ∂ in the long exact cohomology sequence associated to (2).

This theorem is proved in [8]. It is our main device to construct crossed modules for a given 3-cohomology class. In fact, any equivalence class of crossed modules contains a representative of the above form:

Proposition 1. *Let $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$ be a crossed module of Lie algebras. Then there exists a crossed module of the above constructed type*

$$\mu_1 : V_2 \rightarrow \mathfrak{r} = V_3 \times_{\alpha} \mathfrak{g}$$

which is equivalent to $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$.

Proof. To $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$ belongs an exact sequence

$$0 \rightarrow V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \xrightarrow{\pi} \mathfrak{g} \rightarrow 0$$

with $\ker(\mu) =: V$ and $\mathfrak{g} := \operatorname{coker}(\mu)$, and a cohomology class $[\gamma] \in H^3(\mathfrak{g}, V)$ by Gerstenhaber's theorem. Now, embed the \mathfrak{g} -module $V = V_1$ into an injective \mathfrak{g} -module V_2 , and form the quotient \mathfrak{g} -module V_3 yielding an exact sequence

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0.$$

As V_2 is injective, the connecting homomorphism

$$\partial : H^2(\mathfrak{g}, V_3) \rightarrow H^3(\mathfrak{g}, V_1)$$

is an isomorphism, thus the class $[\gamma]$ has a preimage $[\alpha] \in H^2(\mathfrak{g}, V_3)$. By Theorem 3, the data defines a crossed module of the desired type. \square

Having seen that we can always find *some* 2-cocycle α such that the above glued crossed module represents a given cohomology class, the only difficulty in this approach is to find a *meaningful* 2-cocycle α associated to a given crossed module.

Definition 2. *Given an equivalence class of crossed modules $[\mu : \mathfrak{m} \rightarrow \mathfrak{n}] \in \operatorname{crmod}(\mathfrak{g}, V)$ (or, equivalently, a class $[\gamma] \in H^3(\mathfrak{g}, V)$), a crossed module of the form*

$$\mu' : V_2 \rightarrow V_3 \times_{\alpha} \mathfrak{g}$$

for some \mathfrak{g} -modules V_2 and V_3 and some 2-cocycle $\alpha \in Z^2(\mathfrak{g}, V_3)$ which is equivalent to μ is called an abelian representative of the class $[\mu : \mathfrak{m} \rightarrow \mathfrak{n}]$.

2.2 Construction from a central extension of an ideal

Following Neeb [7], we look in this section at a crossed module of Lie algebras as the data of a general extension of Lie algebras

$$0 \rightarrow \mathfrak{l} \xrightarrow{\iota} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \rightarrow 0, \tag{4}$$

together with a central extension

$$0 \rightarrow \mathfrak{z} \rightarrow \widehat{\mathfrak{l}} \xrightarrow{p} \mathfrak{l} \rightarrow 0$$

of the ideal \mathfrak{l} of the extension (4).

If we want that the given general extension and central extension of \mathfrak{l} form a crossed module

$$\mu := \widehat{\mathfrak{l}} \xrightarrow{p} \mathfrak{l} \xrightarrow{\iota} \widehat{\mathfrak{g}},$$

they have to be compatible in the sense that the action of \mathfrak{l} on $\widehat{\mathfrak{l}}$ extends to an action of $\widehat{\mathfrak{g}}$ on $\widehat{\mathfrak{l}}$ satisfying the requirements of Definition 1.

In the following, we will use this slightly different framework to derive once again the cohomology class $[\gamma]$ associated to a crossed module, i.e. in the following, we assume the data of the two extensions together with the extension of the action to form a crossed module. In particular, there is a fixed action of \mathfrak{g} on \mathfrak{z} .

The Lie bracket on the central extension $\widehat{\mathfrak{l}}$ can be given in a very explicit way. Namely, we may assume $\widehat{\mathfrak{l}} = \mathfrak{z} \oplus_{\omega} \mathfrak{l}$, meaning that the bracket reads as

$$[(z_1, l_1), (z_2, l_2)] = (\omega(l_1, l_2), [l_1, l_2]).$$

Here $\omega \in Z^2(\mathfrak{l}, \mathfrak{z})$ is the 2-cocycle defining the central extension. Note that \mathfrak{l} acts trivially on \mathfrak{z} , and the given \mathfrak{g} -action on \mathfrak{z} extends trivially to a $\widehat{\mathfrak{g}}$ -action, still denoted $z \mapsto x \cdot z$.

Now this $\widehat{\mathfrak{g}}$ -action on \mathfrak{z} is supposed to extend to an action on $\widehat{\mathfrak{l}}$, denoted by

$$x \cdot (z, l) = (x \cdot z + \theta(x, l), [x, l]) \quad (5)$$

with some linear map $\theta : \widehat{\mathfrak{g}} \oplus \mathfrak{l} \rightarrow \mathfrak{z}$. Property (b) of Definition 1 implies for $l_1 \in \mathfrak{l} \subset \widehat{\mathfrak{g}}$

$$l_1 \cdot (z, l_2) = [(0, l_1), (z, l_2)] = (\omega(l_1, l_2), [l_1, l_2]),$$

meaning that $\theta|_{\mathfrak{l} \times \mathfrak{l}} = \omega$.

Lemma 1. *Fix an element $x \in \widehat{\mathfrak{g}}$.*

(a) *A linear map $\theta_x \in \text{Hom}(\mathfrak{l}, \mathfrak{z})$ defines a derivation*

$$\zeta(x) : \widehat{\mathfrak{l}} \rightarrow \widehat{\mathfrak{l}}, \quad (z, l) \mapsto (x \cdot z + \theta_x(l), [x, l])$$

if and only if $d^{\mathfrak{l}}\theta_x = x \cdot \omega$, where $d^{\mathfrak{l}}$ is the Lie algebra coboundary operator of \mathfrak{l} with values in $\text{Hom}(\mathfrak{l}, \mathfrak{z})$ and ω is seen as $\omega \in C^1(\mathfrak{l}, \text{Hom}(\mathfrak{l}, \mathfrak{z}))$.

The condition $d^{\mathfrak{l}}\theta_x = x \cdot \omega$ means explicitly for all $l, l' \in \mathfrak{l}$

$$x \cdot \omega(l, l') - \omega([x, l], l') - \omega([l', x], l) + \theta_x([l, l']) = 0.$$

(b) *Suppose that the linear map $\theta : \widehat{\mathfrak{g}} \rightarrow \text{Hom}(\mathfrak{l}, \mathfrak{z}), x \mapsto \theta_x$ satisfies condition (a).*

Then θ defines a representation of $\widehat{\mathfrak{g}}$ on $\widehat{\mathfrak{l}}$ via the formula

$$x \cdot (z, l) = (x \cdot z + \theta_x(l), [x, l])$$

if and only if θ is a 1-cocycle w.r.t. the natural $\widehat{\mathfrak{g}}$ -structure on $C^1(\mathfrak{l}, \mathfrak{z}) = \text{Hom}(\mathfrak{l}, \mathfrak{z})$.

This condition means explicitly for $x, x' \in \widehat{\mathfrak{g}}$ and $l \in \mathfrak{l}$

$$x \cdot \theta(x', l) - x' \cdot \theta(x, l) - \theta([x, x'], l) + \theta(x, [x', l]) + \theta(x', [l, x]) = 0.$$

Proof. see [7]. \square

We saw that the action of $\widehat{\mathfrak{g}}$ on $\widehat{\mathfrak{l}}$ is determined by the map $\theta : \widehat{\mathfrak{g}} \times \mathfrak{l} \rightarrow \mathfrak{z}$. The preceding Lemma states conditions on θ coming from the fact that $\widehat{\mathfrak{g}} \rightarrow \text{End}(\widehat{\mathfrak{l}})$ is a representation (condition (b)) and that it is a representations by derivations, i.e. in fact $\widehat{\mathfrak{g}} \rightarrow \text{der}(\widehat{\mathfrak{l}})$ (condition (a)).

Now we define the 3-cocycle associated in this context to the crossed modules we started with. $\theta|_{\mathfrak{l} \times \mathfrak{l}}$ is alternating, thus we can extend $\theta : \widehat{\mathfrak{g}} \times \mathfrak{l} \rightarrow \mathfrak{z}$ by skewsymmetry to a bilinear map

$$\widetilde{\theta} : \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \rightarrow \mathfrak{z}. \quad (6)$$

Then $d^{\mathfrak{g}}\widetilde{\theta} \in Z^3(\widehat{\mathfrak{g}}, \mathfrak{z})$ is a 3-cocycle which vanishes on $\mathfrak{l} \times \widehat{\mathfrak{g}}^2$ by condition (b) of the Lemma. This means that as soon as the cocycle is evaluated on a triple of elements where at least one comes from \mathfrak{l} , it vanishes. Therefore, it passes to the quotient $\mathfrak{g} = \widehat{\mathfrak{g}}/\mathfrak{l}$ to a 3-cocycle $\chi \in Z^3(\mathfrak{g}, \mathfrak{z})$ with $d^{\mathfrak{g}}\widetilde{\theta} = \pi^*\chi$.

Remark 3. Observe that one advantage of this approach is that here the coboundary operator $d^{\widehat{\mathfrak{g}}}$ refers to a genuine action of $\widehat{\mathfrak{g}}$ on \mathfrak{z} , and we need not deal with formal expressions of coboundary operators.

Lemma 2. The 3-cocycle χ defines a class $[\chi] \in H^3(\mathfrak{g}, \mathfrak{z})$ which does not depend on the choice of the cocycle ω , neither on the choice of the extension $\widetilde{\theta}$ of θ .

Proof. This is shown in [7]. \square

Let us come back to the formula for the characteristic class of a crossed module which we saw in the previous section.

Proposition 2. Let $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$ be a crossed module. Let

$$0 \rightarrow V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \xrightarrow{\pi} \mathfrak{g} \rightarrow 0$$

be the associated four term exact sequence and denote $\mathfrak{l} = \text{im}(\mu)$.

Then the classes $[\gamma]$, where $\gamma \in Z^3(\mathfrak{g}, V)$ is obtained as in Equation (1) by using sections in the four term exact sequence, and $[\chi]$, where $\chi = d^{\mathfrak{g}}\widetilde{\theta}$ is defined in Lemma 2 (with $\mathfrak{z} = V$), coincide in $H^3(\mathfrak{g}, V)$.

Proof. First choose $\omega \in Z^2(\mathfrak{l}, V)$ with $\mathfrak{m} =: \widehat{\mathfrak{l}} = V \oplus_{\omega} \mathfrak{l}$ and a linear section $\rho : \mathfrak{g} \rightarrow \mathfrak{n}$. Associated to ρ , we have for $x, y, z \in \mathfrak{g}$

$$\alpha(x, y) = [\rho(x), \rho(y)] - \rho([x, y]) \in \mathfrak{n},$$

and $\beta(x, y) \in \mathfrak{m}$ with

$$\mu\beta(x, y) = \alpha(x, y).$$

We therefore get $d^{\mathfrak{m}}\beta(x, y, z)$ and $\gamma(x, y, z) \in V$ such that $i\gamma(x, y, z) = d^{\mathfrak{m}}\beta(x, y, z)$ as before. Now as $\beta(x, y) \in \widehat{\mathfrak{l}} = V \oplus_{\omega} \mathfrak{l}$, write

$$\beta = (\beta_V, \alpha) \quad \text{with} \quad \beta_V \in C^2(\mathfrak{g}, V).$$

On the other hand, write the action of \mathfrak{m} on $\hat{\mathfrak{l}}$ as $x \cdot (z, l) = (x \cdot z + \theta(x, l), [x, l])$ with a bilinear map $\theta : \mathfrak{n} \times \mathfrak{l} \rightarrow V$. Then

$$\begin{aligned} d^{\mathfrak{m}}\beta(x, y, z) &= \sum_{\text{cycl.}} \left(\rho(x) \cdot (\beta_V(y, z), \alpha(y, z)) - (\beta_V([x, y], z), \alpha([x, y], z)) \right) \\ &= \sum_{\text{cycl.}} \left((x \cdot \beta_V(y, z) + \theta(\rho(x), \alpha(y, z)), [\rho(x), \alpha(y, z)]) \right. \\ &\quad \left. - (\beta_V([x, y], z), \alpha([x, y], z)) \right). \end{aligned}$$

As $d^{\mathfrak{m}}\beta(x, y, z) \in \ker(\mu)$, the \mathfrak{l} -component of this expression must be zero. This may be seen as an abstract Bianchi identity, cf [7]. We thus obtain

$$d^{\mathfrak{m}}\beta(x, y, z) = d^V\beta_V(x, y, z) + \sum_{\text{cycl.}} \theta(\rho(x), \alpha(y, z)) = \gamma(x, y, z).$$

We must now compare this to $\chi(x, y, z)$. For this, let $\tilde{\theta} \in C^2(\mathfrak{m}, V)$ be an alternating extension of θ . We may assume that $\rho^*\tilde{\theta} = \beta_V$ up to a coboundary. We show now that $\pi^*(d^{\mathfrak{m}}\beta) = d^{\mathfrak{m}}\tilde{\theta}$, so that $[\chi] = [d^{\mathfrak{m}}\beta] \in H^3(\mathfrak{g}, V)$ as claimed.

$$\begin{aligned} d^{\mathfrak{m}}\tilde{\theta}(x, y, z) &= \sum_{\text{cycl.}} \left(x \cdot \tilde{\theta}(\rho(y), \rho(z)) - \tilde{\theta}([\rho(x), \rho(y)], \rho(z)) \right) \\ &= \sum_{\text{cycl.}} \left(x \cdot \tilde{\theta}(\rho(y), \rho(z)) - \tilde{\theta}(\rho([x, y]) + \alpha(x, y), \rho(z)) \right) \\ &= \sum_{\text{cycl.}} \left(x \cdot \beta_V(y, z) - \beta_V([x, y], z) - \tilde{\theta}(\alpha(x, y), \rho(z)) \right) \\ &= \sum_{\text{cycl.}} \left(x \cdot \beta_V(y, z) - \beta_V([x, y], z) + \theta(\rho(x), \alpha(y, z)) \right) \\ &= d^{\mathfrak{m}}\beta_V(x, y, z) + \sum_{\text{cycl.}} \theta(\rho(x), \alpha(y, z)) \\ &= d^{\mathfrak{m}}\beta(x, y, z), \end{aligned}$$

where we used the definition of α passing from the first to the second line, then the assumption $\rho^*\tilde{\theta} = \beta_V$, then the fact that $\tilde{\theta}$ reduces to θ in case one variable is in \mathfrak{l} , and finally the relation between $d^{\mathfrak{m}}\beta$ and $d^{\mathfrak{m}}\beta_V$ we obtained earlier. \square

3 The string Lie algebra

The *string group* arises in the following context. Recall that for a connected, simply connected, finite dimensional, semi-simple Lie group G , $\pi_3(G)$ counts the number of simple factors of G . Starting with an arbitrary topological group, the connected component of $1 \in K$, denoted K_1 , has $\pi_i(K) = \pi_i(K_1)$ for all $i > 0$, but $\pi_0(K_1) = 0$. In the same way (in case K is, e.g. locally contractible and

connected), the universal covering \tilde{K} of K , satisfies $\pi_i(\tilde{K}) = \pi_i(K)$ for all $i > 1$, but $\pi_1(\tilde{K}) = \pi_0(\tilde{K}) = 0$. All finite dimensional Lie groups K have $\pi_2(K) = 0$. The string group $\text{Str}(G)$ of G is defined in the same way as being a group S such that $\pi_i(S) = \pi_i(G)$ for all $i > 3$, but $\pi_3(S) = \pi_2(S) = \pi_1(S) = \pi_0(S) = 0$. By what we remarked before, the group S cannot be a finite dimensional Lie group, because its maximal compact subgroup (which is homotopy equivalent to S) should contain a simple factor. The string group $\text{Str}(G)$ for a connected, simply connected, finite dimensional, semi-simple Lie group G has many homotopy theoretical realizations. We will focuss here on the Lie algebraic version given in [7] of it.

Let \mathfrak{g} be a finite dimensional real Lie algebra and denote by I the closed interval $I = [0, 1]$. We consider the *smooth path algebra*

$$P(\mathfrak{g}) := \{\xi \in C^\infty(I, \mathfrak{g}) \mid \xi(0) = 0\}$$

of \mathfrak{g} . Then evaluation in $1 \in I$, denoted ev_1 , leads to a short exact sequence

$$0 \rightarrow \mathfrak{l} \rightarrow P(\mathfrak{g}) \rightarrow \mathfrak{g} \rightarrow 0,$$

where $\mathfrak{l} := \ker(\text{ev}_1)$ is the ideal of closed paths in $P(\mathfrak{g})$ and a linear section $\rho : \mathfrak{g} \rightarrow P(\mathfrak{g})$ is given by $\rho(x)(t) := tx$. Note that \mathfrak{l} is larger than the Lie algebra $C^\infty(S^1, \mathfrak{g})$ which corresponds to those elements of \mathfrak{l} for which all derivatives have the same boundary value in 0 and 1.

Let $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{z}$ be an invariant bilinear form. The invariance condition means that for all $x, y, z \in \mathfrak{g}$, we have

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

We consider in the following \mathfrak{z} as a trivial $P(\mathfrak{g})$ -module. Then the Lie algebra \mathfrak{l} has a central extension $\hat{\mathfrak{l}} := \mathfrak{z} \times_\omega \mathfrak{l}$ where the cocycle ω is given by

$$\omega(\xi, \eta) := \int_0^1 \kappa(\xi, \eta') := \int_0^1 \kappa(\xi, \eta')(t) dt.$$

We define $\tilde{\omega} \in C^2(P(\mathfrak{g}), \mathfrak{z})$ by

$$\begin{aligned} \tilde{\omega}(\xi, \eta) &:= \frac{1}{2} \int_0^1 (\kappa(\xi, \eta') - \kappa(\eta, \xi')) = \frac{1}{2} \int_0^1 (2\kappa(\xi, \eta') - \kappa(\eta, \xi')) = \\ &= \int_0^1 \kappa(\xi, \eta') - \frac{1}{2} \kappa(\xi, \eta)(1). \end{aligned}$$

We observe that for $(\xi, \eta) \in P(\mathfrak{g}) \times \mathfrak{l}$, we have $\tilde{\omega}(\xi, \eta) = \theta(\xi, \eta) := \int_0^1 \kappa(\xi, \eta')$. This is the map θ which we used in Equation (5) in order to associate a 3-class to a crossed module. It is therefore clear that $\tilde{\omega}$ may serve as $\tilde{\theta}$, the skewsymmetric extension of θ to $P(\mathfrak{g}) \times P(\mathfrak{g})$.

For the following construction of the crossed module related to the string Lie algebra, we compute:

$$\begin{aligned}
\sum_{\text{cycl.}} \int_0^1 \kappa([\xi, \eta], \zeta') &= \int_0^1 \kappa([\xi, \eta], \zeta') + \kappa([\eta, \zeta], \xi') + \kappa([\zeta, \xi], \eta') = \\
&= \int_0^1 \kappa([\xi, \eta], \zeta') + \kappa([\xi', \eta], \zeta) + \kappa([\xi, \eta'], \zeta) = \\
&= \int_0^1 \kappa([\xi, \eta], \zeta)' = \kappa([\xi, \eta], \zeta)(1).
\end{aligned}$$

This implies

$$\begin{aligned}
\sum_{\text{cycl.}} \int_0^1 \kappa([\xi, \eta]', \zeta) &= \sum_{\text{cycl.}} \int_0^1 \kappa([\xi', \eta], \zeta) + \kappa([\xi, \eta'], \zeta) = \\
&= \sum_{\text{cycl.}} \int_0^1 \kappa(\xi', [\eta, \zeta]) + \kappa([\zeta, \xi], \eta') = \\
&= \kappa([\eta, \zeta], \xi)(1) + \kappa([\zeta, \xi], \eta)(1) = \\
&= 2\kappa([\eta, \zeta], \xi)(1) = 2\kappa([\xi, \eta], \zeta)(1).
\end{aligned}$$

Here we used invariance of κ and skewsymmetry of $[\cdot, \cdot]$ to rearrange the three terms to a complete derivative. We deduce from this computation

$$\begin{aligned}
(d^{P(\mathfrak{g})}\tilde{\omega})(\xi, \eta, \zeta) &= \frac{1}{2} \int_0^1 \sum_{\text{cycl.}} \left(\kappa(\xi, [\eta, \zeta]') - \kappa([\eta, \zeta], \xi') \right) \\
&= \frac{1}{2} \left(2\kappa([\eta, \zeta], \xi)(1) - \kappa([\eta, \zeta], \xi)(1) \right) = \frac{1}{2} \kappa([\eta, \zeta], \xi)(1),
\end{aligned}$$

and the expression $d^{P(\mathfrak{g})}\tilde{\omega}$ thus vanishes on $(P(\mathfrak{g}))^2 \times \mathfrak{l}$. This is (by restriction of the arguments) the cocycle identity for the cochain $\theta \in C^1(P(\mathfrak{g}), \mathfrak{z})$ and the cochain $\theta \in C^0(P(\mathfrak{g}), \text{Hom}(\mathfrak{l}, \mathfrak{z})) = \text{Hom}(\mathfrak{l}, \mathfrak{z})$.

By Lemma 1, the formula

$$x \cdot (z, l) = (\theta(x, l), [x, l])$$

thus defines a representation of $P(\mathfrak{g})$ on \mathfrak{l} . Moreover, the map $\mu : \hat{\mathfrak{l}} \rightarrow P(\mathfrak{g})$ defines a crossed module, which we call the *string Lie algebra crossed module*. By our computations, we have computed the 3-cocycle χ defined by $\text{ev}_1^* \chi = d^{P(\mathfrak{g})}\tilde{\omega}$. We have

$$\chi(x, y, z) := \frac{1}{2} \kappa([x, y], z),$$

and $\chi \in Z^3(\mathfrak{g}, \mathfrak{z})$. Up to the factor $\frac{1}{2}$, the cocycle χ is called the *Cartan cocycle*. The cocycle χ represents the class in $H^3(\mathfrak{g}, \mathfrak{z})$ of the crossed module $\mu : \hat{\mathfrak{l}} \rightarrow P(\mathfrak{g})$. In case $\kappa \neq 0$, the class $[\chi] \in H^3(\mathfrak{g}, \mathfrak{z})$ is non zero. For a simple Lie algebra \mathfrak{g} , taking κ to be the *Killing form*, the 1-dimensional cohomology space $H^3(\mathfrak{g}, \mathbb{R})$ is generated by the class $[\chi]$.

Remark 4. *This example should be compared to the example of the string group after Baez-Crans-Schreiber-Stevenson in [1]. The string Lie algebra here is the Lie algebra crossed module corresponding to the Lie group crossed module given by the string group.*

In the following, our aim is to construct an *abelian representative* of the equivalence class of the crossed module associated to the string Lie algebra. This will play a role in a later section for the construction of quasi-invariant tensors.

4 Lifting 3-cocycles with values in \mathbb{C}

4.1 Preliminaries

We denote by $U\mathfrak{g}$ the universal enveloping algebra of a Lie algebra \mathfrak{g} , and by $U\mathfrak{g}^+$ the kernel of the augmentation map

$$\varepsilon : U\mathfrak{g} \rightarrow \mathbb{C}$$

Identifying \mathbb{C} with its dual, this gives rise to a short exact sequence of (left) \mathfrak{g} -modules:

$$0 \rightarrow \mathbb{C} \rightarrow (U\mathfrak{g})^\vee \rightarrow (U\mathfrak{g}^+)^\vee \rightarrow 0 \quad (7)$$

Recall that $U\mathfrak{g}$ is a cocommutative Hopf algebra. In particular, its vector space of endomorphisms $\text{End}_{\mathbb{C}}(U\mathfrak{g})$ is equipped with the so called *convolution product* that we denote \star . The n -fold iterated coproduct of an element x of $U\mathfrak{g}$ will be written

$$\sum_{(x)} x^{(1)} \otimes \cdots \otimes x^{(n)} \in U\mathfrak{g}^{\otimes n}$$

in Sweedler's notation.

$\mathbb{C}[t]$ denotes the \mathbb{C} -algebra of polynomials in the variable t with complex coefficients. If λ is a complex number, and V is a vector space, we have an evaluation map $\text{ev}_\lambda : V \otimes \mathbb{C}[t] \rightarrow V$ which sends $v \otimes t^n$ to $\lambda^n v$, for all v in V . In particular, every $\mathbb{C}[t]$ -linear map $B : V \otimes \mathbb{C}[t] \rightarrow V \otimes \mathbb{C}[t]$ is uniquely determined by its \mathbb{C} -linear restriction to V : $B_t : V \rightarrow V \otimes \mathbb{C}[t]$. We will employ the notation B_λ to denote the composite $\text{ev}_\lambda \circ B_t : V \rightarrow V$.

4.2 Using the contracting homotopy to lift cocycles

Let \mathfrak{g} be a Lie algebra over \mathbb{C} with Lie bracket $[-, -]$, and denote by $\eta : \mathbb{C} \rightarrow U\mathfrak{g}$ the unit map.

Definition 3. *If x is an element of $U\mathfrak{g}$ and g is an element of \mathfrak{g} , then we set*

$$(a) \text{ pr}(x) := \sum_{k \geq 0} \frac{(-1)^k}{k+1} (\text{Id} - \eta\epsilon)^{\star k}, \text{ and this defines an endomorphism pr of } U\mathfrak{g},$$

- (b) $\phi_t(x) := \sum_{n \geq 0} \frac{1}{n!} \text{pr}^{\star n}(x)$, which defines an element of $U\mathfrak{g} \otimes \mathbb{C}[t]$ and thus a $\mathbb{C}[t]$ -linear endomorphism of $U\mathfrak{g} \otimes \mathbb{C}[t]$,
- (c) $A_t(x, g) := \sum_{(x)} \phi_{-t}(x^{(1)}) \phi_t(x^{(2)}) g$.

The following proposition summarizes the different properties of pr , ϕ_t and A_t we will need in the sequel.

Proposition 3. *Let Δ denote the unique $\mathbb{C}[t]$ -linear coproduct of $U\mathfrak{g} \otimes \mathbb{C}[t]$ extending the comultiplication on $U\mathfrak{g}$. Then,*

1. *pr is idempotent ($\text{pr} \circ \text{pr} = \text{pr}$) takes its values in $\mathfrak{g} \subset U\mathfrak{g}$. More precisely, if g_1, \dots, g_n are elements of \mathfrak{g} , then*

$$\text{pr}(g_1 \cdots g_n) = \frac{1}{n^2} \sum_{\sigma \in \Sigma_n} (-1)^{d(\sigma)} \binom{n-1}{d(\sigma)}^{-1} [g_{\sigma(1)}, \dots, g_{\sigma(n)}] \quad (8)$$

where the notation $[h_1, \dots, h_n]$ stands for the iterated Lie bracket

$$[h_1, [h_2, [\dots, [h_{n-1}, h_n] \dots]]]$$

of elements h_1, \dots, h_n of \mathfrak{g} .

2. *ϕ_t is an endomorphism of coalgebra i.e.*

$$\Delta(\phi_t(x)) = (\phi_t \otimes \phi_t) \Delta(x) = \sum_{(x)} \phi_t(x^{(1)}) \otimes \phi_t(x^{(2)})$$

for all x in $U\mathfrak{g} \subset U\mathfrak{g} \otimes \mathbb{C}[t]$.

3. $\phi_0 = \eta\epsilon$ and $\phi_1 = \text{Id}$,
4. $\phi_t \star \phi_{-t} = \eta\epsilon$ on $U\mathfrak{g} \subset U\mathfrak{g} \otimes \mathbb{C}[t]$.
5. for all x in $U\mathfrak{g}$ and g in \mathfrak{g} , $A_t(x, g)$ is an element of \mathfrak{g} and

$$A_t(x, g) = - \sum_{(x)} \phi_{-t}(x^{(1)}) g \phi_t(x^{(2)})$$

Let $f : \Lambda^3 \mathfrak{g} \rightarrow \mathbb{C}$ be a Chevalley–Eilenberg 3-cochain of \mathfrak{g} with values in \mathbb{C} . Using the identification

$$C^2(\mathfrak{g}; (U\mathfrak{g})^\vee) \cong C_2(\mathfrak{g}; U\mathfrak{g})^\vee$$

we can use f to produce a 2-cochain α in $C^2(\mathfrak{g}; (U\mathfrak{g})^\vee)$ by setting:

$$\tilde{\alpha}(x, g_1, g_2) := \sum_{(x)} \int_0^1 dt f(\text{pr}(x^{(1)}), A_t(x^{(2)}, g_1), A_t(x^{(3)}, g_2)) \quad (9)$$

for all x in $U\mathfrak{g}$, g_1 and g_2 in \mathfrak{g} . This 2-cochain $\tilde{\alpha}$ induces a 2-cochain α in $C^2(\mathfrak{g}; (U\mathfrak{g}^+)^\vee)$ by restriction to $U\mathfrak{g}^+ \subset U\mathfrak{g}$.

Example 1. If $x = \lambda$ is an element of $\mathbb{C} \subset U\mathfrak{g}$ then formula (9) reads

$$\tilde{\alpha}(\lambda, g_1, g_2) = 0$$

If $x = g$ is an element of $\mathfrak{g} \subset U\mathfrak{g}$, then formula (9) reads

$$\alpha(g, g_1, g_2) = \int_0^1 dt f(g, tg_1, tg_2) = \frac{1}{3}f(g, g_1, g_2)$$

If $x = gh$ is a product of two elements of \mathfrak{g} , then formula (9) reads

$$\begin{aligned} \alpha(gh, g_1, g_2) &= \int_0^1 dt \left(\frac{1}{2}f([g, h], tg_1, tg_2) + f(g, \frac{(t^2 - t)}{2}[h, g_1], tg_2) \right. \\ &\quad + f(g, tg_1, \frac{(t^2 - t)}{2}[h, g_2]) + f(h, \frac{(t^2 - t)}{2}[g, g_1], tg_2) \\ &\quad \left. + f(h, tg_1, \frac{(t^2 - t)}{2}[g, g_2]) \right) \\ &= \frac{1}{12} (2f([g, h], g_1, g_2) - f(g, [h, g_1], g_2) - f(g, g_1, [h, g_2]) \\ &\quad - f(h, [g, g_1], g_2) - f(h, g_1, [g, g_2])) \end{aligned}$$

Proposition 4. If f is a 3-cocycle with values in \mathbb{C} , then the restriction of $d_{CE}(\tilde{\alpha})$ to $\mathbb{C} \otimes \Lambda^3 \mathfrak{g} \subset U\mathfrak{g} \otimes \Lambda^3 \mathfrak{g}$ is f . In particular, α is a 2-cocycle in $C^2(\mathfrak{g}; U\mathfrak{g}^+)$ and $[f] = \partial[\alpha]$ in $H^3(\mathfrak{g}; \mathbb{C})$, where ∂ is the connecting homomorphism of the long exact sequence

$$\cdots \rightarrow H^2(\mathfrak{g}; \mathbb{C}) \rightarrow H^2(\mathfrak{g}; (U\mathfrak{g})^\vee) \rightarrow H^2(\mathfrak{g}; (U\mathfrak{g}^+)^\vee) \xrightarrow{\partial} H^3(\mathfrak{g}; \mathbb{C}) \rightarrow H^3(\mathfrak{g}; (U\mathfrak{g})^\vee) \rightarrow \cdots \quad (10)$$

associated to the short exact sequence of coefficients (7).

Proof. This assertion is a consequence of a more general statement asserting that the right hand side of formula (9) is just the degree 3 component of the dual s_*^\vee of certain contracting homotopy s_* of $(C_*(\mathfrak{g}; U\mathfrak{g}), d_{CE})$ applied to f . However, for the sake of completeness, let us reprove this particular case here: For x in $U\mathfrak{g}$ and g_1, g_2, g_3 in \mathfrak{g} , we have

$$\begin{aligned} d_{CE}(\alpha)(x, g_1, g_2, g_3) &= \alpha(xg_1, g_2, g_3) - \alpha(xg_2, g_1, g_3) + \alpha(xg_3, g_1, g_2) \\ &\quad + \alpha(x, [g_1, g_2], g_3) - \alpha(x, [g_1, g_3], g_2) + \alpha(x, g_1, [g_2, g_3]) \end{aligned}$$

Using the Hopf relation on $U\mathfrak{g}$ and the fact that the g_i 's are primitive, the

first part of the preceeding equality reads

$$\begin{aligned} \alpha(xg_1, g_2, g_3) - \alpha(xg_2, g_1, g_3) + \alpha(xg_3, g_1, g_2) &= \sum_{(x)} \int_0^1 dt \\ & f(\text{pr}(x^{(1)}g_1), A_t(x^{(2)}g_2), A_t(x^{(3)}g_3)) + f(\text{pr}(x^{(1)}), A_t(x^{(2)}g_1, g_2), A_t(x^{(3)}g_3)) + \\ & + f(\text{pr}(x^{(1)}), A_t(x^{(2)}g_2), A_t(x^{(3)}g_1, g_3)) - f(\text{pr}(x^{(1)}g_2), A_t(x^{(2)}g_1), A_t(x^{(3)}g_3)) - \\ & f(\text{pr}(x^{(1)}), A_t(x^{(2)}g_2, g_1), A_t(x^{(3)}g_3)) - f(\text{pr}(x^{(1)}), A_t(x^{(2)}g_1), A_t(x^{(3)}g_2, g_3)) \\ & + f(\text{pr}(x^{(1)}g_3), A_t(x^{(2)}g_1), A_t(x^{(3)}g_2)) + f(\text{pr}(x^{(1)}), A_t(x^{(2)}g_3, g_1), A_t(x^{(3)}g_2)) \\ & + f(\text{pr}(x^{(1)}), A_t(x^{(2)}g_1), A_t(x^{(3)}g_3, g_2)) \end{aligned}$$

But for y in $U\mathfrak{g}$ and g, h in \mathfrak{g} , properties 4. and 5. of ϕ_t and A_t listed in Proposition 3 imply that

$$\begin{aligned} A_t(yg, h) - A_t(yh, g) &= \sum_{(y)} \phi_{-t}(y^{(1)}g)\phi_t(y^{(2)}h) - \phi_{-t}(y^{(1)}h)\phi_t(y^{(2)}g) + \\ & + \phi_{-t}(y^{(1)})\phi_t(y^{(2)}gh) - \phi_{-t}(y^{(1)})\phi_t(y^{(2)}hg) \\ &= \sum_{(y)} \phi_{-t}(y^{(1)}g)\phi_t(y^{(2)}h) - \phi_{-t}(y^{(1)}h)\phi_t(y^{(2)}g) + A_t(y, [g, h]) \\ &= \sum_{(y)} \phi_{-t}(y^{(1)}g)\phi_t(y^{(2)})\phi_{-t}(y^{(3)})\phi_t(y^{(4)}h) - \\ & \phi_{-t}(y^{(1)}h)\phi_t(y^{(2)})\phi_{-t}(y^{(3)})\phi_t(y^{(4)}g) + A_t(y, [g, h]) \\ &= A_t(y, [g, h]) - \sum_{(y)} [A_t(y^{(1)}, g), A_t(y^{(2)}, h)] \end{aligned}$$

Thus, all terms of the form $f(\cdots, A_t(x^{(i)}g_j, g_k), \cdots)$ appearing in $\alpha(xg_1, g_2, g_3) - \alpha(xg_2, g_1, g_3) + \alpha(xg_3, g_1, g_2)$ can be sorted by pairs (j, k) to cancel the corresponding term of the form $\alpha(\cdots, [g_j, g_k], \cdots)$ in $d_{CE}(\alpha)(x, g_1, g_2, g_3)$. Hence, we are left with

$$\begin{aligned} d_{CE}(\alpha)(x, g_1, g_2, g_3) &= \sum_{(x)} \int_0^1 dt f(\text{pr}(x^{(1)}g_1), A_t(x^{(2)}g_2), A_t(x^{(3)}g_3)) - \\ & f(\text{pr}(x^{(1)}), [A_t(x^{(2)}g_1), A_t(x^{(3)}g_2)], A_t(x^{(4)}g_3)) - f(\text{pr}(x^{(1)}g_2), A_t(x^{(2)}g_1), A_t(x^{(3)}g_3)) \\ & + f(\text{pr}(x^{(1)}), A_t(x^{(2)}g_1), [A_t(x^{(3)}g_2), A_t(x^{(4)}g_3)]) + f(\text{pr}(x^{(1)}g_3), A_t(x^{(2)}g_1), A_t(x^{(3)}g_2)) \\ & + f(\text{pr}(x^{(1)}), [A_t(x^{(2)}g_1), A_t(x^{(3)}g_3)], A_t(x^{(4)}g_2)) \end{aligned}$$

Since f is a cocycle, we have that

$$d_{CE}(f)(\text{pr}(x^{(1)}), A_t(x^{(2)}g_1), A_t(x^{(3)}g_2), A_t(x^{(4)}g_3)) = 0$$

which enables to rewrite the three terms of $d_{CE}(\alpha)(x, g_1, g_2, g_3)$ involving Lie brackets to obtain

$$\begin{aligned} d_{CE}(\alpha)(x, g_1, g_2, g_3) &= \sum_{(x)} \int_0^1 dt f(\text{pr}(x^{(1)} g_1), A_t(x^{(2)}, g_2), A_t(x^{(3)}, g_3)) - \\ &\quad f([\text{pr}(x^{(1)}), A_t(x^{(2)}, g_1)], A_t(x^{(3)}, g_2), A_t(x^{(4)}, g_3)) - f(\text{pr}(x^{(1)} g_2), A_t(x^{(2)}, g_1), A_t(x^{(3)}, g_3)) \\ &\quad + f([\text{pr}(x^{(1)}), A_t(x^{(2)}, g_2)], A_t(x^{(3)}, g_1), A_t(x^{(4)}, g_3)) + f(\text{pr}(x^{(1)} g_3), A_t(x^{(2)}, g_1), A_t(x^{(3)}, g_2)) - \\ &\quad f([\text{pr}(x^{(1)}), A_t(x^{(2)}, g_3)], A_t(x^{(3)}, g_1), A_t(x^{(4)}, g_2)) \end{aligned}$$

But one easily checks that for all y in $U\mathfrak{g}$ and g in \mathfrak{g}

$$\text{pr}(yg) - \sum_{(y)} [\text{pr}(y^{(1)}), A_t(y^{(2)}, g)] = \frac{d}{dt} A_t(y, g)$$

so that, using point 3. of Proposition 3

$$\begin{aligned} d_{CE}(\alpha)(x, g_1, g_2, g_3) &= \sum_{(x)} \int_0^1 dt f\left(\frac{d}{dt} A_t(x^{(1)}, g_1), A_t(x^{(2)}, g_2), A_t(x^{(3)}, g_3)\right) + \\ &\quad f\left(A_t(x^{(1)}, g_1), \frac{d}{dt} A_t(x^{(2)}, g_2), A_t(x^{(3)}, g_3)\right) + f\left(A_t(x^{(1)}, g_1), A_t(x^{(2)}, g_2), \frac{d}{dt} A_t(x^{(3)}, g_3)\right) \\ &= \sum_{(x)} \int_0^1 dt \frac{d}{dt} f(A_t(x^{(1)}, g_1), A_t(x^{(2)}, g_2), A_t(x^{(3)}, g_3)) \\ &= \sum_{(x)} f(A_1(x^{(1)}, g_1), A_1(x^{(2)}, g_2), A_1(x^{(3)}, g_3)) - f(A_0(x^{(1)}, g_1), A_0(x^{(2)}, g_2), A_0(x^{(3)}, g_3)) \\ &= \epsilon(x) f(g_1, g_2, g_3) \end{aligned}$$

from which the proposition follows immediately. \square

Together with Theorem 3, this shows:

Corollary 1. *If f is a 3-cocycle, then the Yoneda product*

$$0 \rightarrow \mathbb{C} \rightarrow (U\mathfrak{g})^\vee \rightarrow (U\mathfrak{g}^+)^\vee \oplus_\alpha \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow 0$$

is a crossed module of Lie algebras whose equivalence class represents $[f]$.

In other words, we have constructed abelian representatives (of the equivalence class of crossed modules associated) to each 3-cohomology class given by some constant-valued 3-cocycle f .

4.3 The example of the Cartan cocycle of $\mathfrak{sl}_2(\mathbb{C})$

In this subsection, $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ and $f : \Lambda^3 \mathfrak{g} \rightarrow \mathbb{C}$ is the Cartan 3-cocycle defined by

$$f(g_1, g_2, g_3) := \kappa(g_1, [g_2, g_3])$$

for all g_1, g_2 and g_3 in \mathfrak{g} , where κ is the Killing form of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (In fact, f is, up to a non-zero scalar, the only non-trivial possible 3-cochain). Our goal is to study the 2-cocycle α lifting f in order to get a concrete description of the crossed module encoding the Cartan cocycle. More precisely, we want to determine the values of

$$\alpha(x, g_1, g_2) := \int_0^1 dt \kappa(\text{pr}(x^{(1)}), [A_t(x^{(2)}, g_1), A_t(x^{(3)}, g_2)])$$

when g_1, g_2 run over a given basis of \mathfrak{g} and x runs over the associated PBW basis of $U\mathfrak{g}^+$. Denote by (X, Y, H) the standard ordered basis of $\mathfrak{sl}_2(\mathbb{C})$ which satisfies the relations

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

In this basis, the problem amounts to determine the values of the following complex numbers

$$B_{XY}(a, b, c) := \alpha(X^a Y^b H^c, X, Y)$$

and

$$B_{XH}(a, b, c) := \alpha(X^a Y^b H^c, X, H)$$

and

$$B_{YH}(a, b, c) := \alpha(X^a Y^b H^c, Y, H)$$

for all natural numbers a, b and c .

If obtaining a general formula for the values of B_{XY} , B_{XH} and B_{YH} seems out of reach for the moment, in particular because the combinatorics appearing in the expression (8) of $\text{pr}(x)$, for a word x of $U\mathfrak{g}$, get more and more complicated as the length of x increases, we can at least show that these values are “often” 0.

Proposition 5. *Let a, b and c be three natural numbers.*

1. *If $a \neq b$, then*

$$B_{XY}(a, b, c) = 0$$

2. *If $a \neq b - 1$, then*

$$B_{XH}(a, b, c) = 0$$

3. *If $a \neq b + 1$, then*

$$B_{YH}(a, b, c) = 0$$

Before entering the proof of the proposition, let us establish the following lemma which shows that non-trivial iterated brackets in $\mathfrak{sl}_2(\mathbb{C})$ must have a very constrained form.

Lemma 3. Let (X_1, \dots, X_n) be a sequence of basis vectors in $\{X, Y, H\}^n$ and denote by $a := \text{card}\{i / X_i = X\}$ and $b := \text{card}\{i / X_i = Y\}$ respectively the number of X 's and the number of Y 's appearing in this sequence. If the iterated bracket

$$[X_1, \dots, X_n] := [X_1, [X_2, [\dots, [X_{n-1}, X_n] \dots]]$$

is not zero, then

$$|a - b| \leq 1$$

In this case, there exists a non-zero complex number λ such that

$$[X_1, \dots, X_n] = \begin{cases} \lambda H & \text{if } a = b, \\ \lambda X & \text{if } a = b + 1, \\ \lambda Y & \text{if } a = b - 1. \end{cases}$$

Proof of the lemma. Let (X_1, \dots, X_n) be a sequence of basis vectors such that $[X_1, \dots, X_n] \neq 0$. We can assume that none of the X_i 's is equal to H since the value of an iterated bracket containing an H is equal to twice the value of the iterated bracket obtained by erasing this H i.e.

$$[Z_1, \dots, Z_i, H, Z_{i+1}, \dots, Z_m] = 2[Z_1, \dots, Z_m]$$

for all Z_1, \dots, Z_m in $\{X, Y, H\}$. We now proceed by induction on the length n of the sequence (X_1, \dots, X_n) . Assume that the proposition holds for all values of k strictly lower than a given n , in particular for $n = n - 2$. Then the non-zero iterated bracket we are interested in takes the form

$$[X_1, [X_2, [X_3, \dots, X_n]]] \neq 0$$

If we denote by a' (resp. b') the number of X 's (resp. the number of Y 's) in the sequence (X_3, \dots, X_n) , applying the conclusion of the proposition (for $k = n - 2$) to the necessarily non-zero bracket $[X_3, \dots, X_n]$ allows to distinguish 3 cases

- $[X_3, \dots, X_n] = \lambda' H$ for some non zero complex number λ' . This implies that $a' = b'$. In addition, we see that in order to have

$$0 \neq [X_1, [X_2, [X_3, \dots, X_n]]] = \lambda' [X_1, [X_2, H]] = 2\lambda' \pm [X_1, X_2]$$

X_1 and X_2 must be different. Hence $a = a' + 1 = b' + 1 = b$ i.e. $a = b$ and $[X_1, [X_2, [X_3, \dots, X_n]]] = \lambda H$ with $\lambda = \pm 2\lambda'$.

- $[X_3, \dots, X_n] = \lambda' X$ for some non zero complex number λ' . This corresponds to the case $a' = b' + 1$. In this case, we see that the only possible choice of X_1 and X_2 leading to a non zero value of $[X_1, [X_2, [X_3, \dots, X_n]]]$ is given by taking $(X_1, X_2) = (X, Y)$, which leads to

$$[X_1, \dots, X_n] = 2\lambda' X \quad \text{and} \quad a = a' + 1 = b' + 2 = b + 1$$

or taking $(X_1, X_2) = (Y, Y)$ for which

$$[X_1, \dots, X_n] = -2\lambda' Y \quad \text{and} \quad a = a' = b' + 1 = b - 1$$

In both cases, the conclusion of the lemma holds.

- $[X_3, \dots, X_n] = \lambda' Y$ for some non zero complex number λ' . In this case, $a' = b' - 1$ and the only possible choices for (X_1, X_2) are $(X_1, X_2) = (Y, X)$, which leads to

$$[X_1, \dots, X_n] = 2\lambda' Y \quad \text{and} \quad a = b - 1$$

and $(X_1, X_2) = (X, X)$, for which

$$[X_1, \dots, X_n] = -2\lambda' X \quad \text{and} \quad a = b + 1$$

Again, we see that theses values of a, b and $[X_1, \dots, X_n]$ satisfy the conclusion of the lemma.

The initialization of the induction is left to the reader. \square

Proof of proposition 5. In view of formula (9) defining α and of the definition of the coproduct on $U\mathfrak{g}$, it is clear that $B_{XY}(a, b, c) := \alpha(X^a Y^b H^c, X, Y)$ can be written as a sum of the form

$$B_{XY}(a, b, c) = \sum_{\bar{a}, \bar{b}, \bar{c}} \lambda_{\bar{a}, \bar{b}, \bar{c}} f(\text{pr}(X^{a_1} Y^{b_1} H^{c_1}), A_t(X^{a_2} Y^{b_2} H^{c_2}, X), A_t(X^{a_3} Y^{b_3} H^{c_3}, Y))$$

where the summation runs over all triples of integers $\bar{a} := (a_1, a_2, a_3)$, $\bar{b} := (b_1, b_2, b_3)$ and $\bar{c} := (c_1, c_2, c_3)$ such that $a_1 + a_2 + a_3 = a$, $b_1 + b_2 + b_3 = b$ and $c_1 + c_2 + c_3 = c$ and where the $\lambda_{\bar{a}, \bar{b}, \bar{c}}$'s are complex numbers. Our aim is to determine which terms of this sum have a chance to be non-zero. For each choice of triples \bar{a} , \bar{b} and \bar{c} , formula (8) and the lemma imply that $\text{pr}(X^{a_1} Y^{b_1} H^{c_1})$ can be non-zero only when $|a_1 - b_1| \leq 1$ and in this case

$$\text{pr}(X^{a_1} Y^{b_1} H^{c_1}) = \begin{cases} \lambda H & \text{if } a_1 = b_1, \\ \lambda X & \text{if } a_1 = b_1 + 1, \\ \lambda Y & \text{if } a_1 = b_1 - 1. \end{cases} \quad (11)$$

for some complex number λ . Moreover, one can check that by definition of the operator A_t , $A_t(X^{a_2} Y^{b_2} H^{c_2}, X)$ is a linear combination of words of length $a_1 + a_2 + c_2 + 1$ containing exactly $a_2 + 1$ copies of X , b_2 copies of Y and c_2 copies of H . Since it is in \mathfrak{g} by proposition 3, it is equal to its own image under pr i.e.

$$A_t(X^{a_2} Y^{b_2} H^{c_2}, X) = \text{pr}(A_t(X^{a_2} Y^{b_2} H^{c_2}, X))$$

Thus, the lemma implies that $A_t(X^{a_2} Y^{b_2} H^{c_2}, X)$ can be non zero only when $|a_2 + 1 - b_2| \leq 1$ and in this case

$$A_t(X^{a_2} Y^{b_2} H^{c_2}, X) = \begin{cases} \beta H & \text{if } a_2 = b_2 - 1, \\ \beta X & \text{if } a_2 = b_2, \\ \beta Y & \text{if } a_2 = b_2 + 1. \end{cases} \quad (12)$$

for some complex number β . Similarly, one can prove that the only non-zero possible values of $A_t(X^{a_3} Y^{b_3} H^{c_3}, Y)$ are given by

$$A_t(X^{a_3} Y^{b_3} H^{c_3}, Y) = \begin{cases} \mu H & \text{if } a_3 = b_3 + 1, \\ \mu X & \text{if } a_3 = b_3 + 2, \\ \mu Y & \text{if } a_3 = b_3. \end{cases} \quad (13)$$

for some complex number μ . Since f is skew-symmetric, the term

$$f(\text{pr}(X^{a_1}Y^{b_1}H^{c_1}), A_t(X^{a_2}Y^{b_2}H^{c_2}, X), A_t(X^{a_3}Y^{b_3}H^{c_3}, Y))$$

can be non-zero only when its three arguments are all different. Equations (11), (12) and (13) show that this can happen only if

$$\begin{aligned} & (a_1, a_2, a_3) = (b_1, b_2, b_3) \\ \text{or } & (a_1, a_2, a_3) = (b_1, b_2 - 2, b_3 + 2) \\ \text{or } & (a_1, a_2, a_3) = (b_1 + 1, b_2 - 1, b_3) \\ \text{or } & (a_1, a_2, a_3) = (b_1 + 1, b_2 - 2, b_3 + 1) \\ \text{or } & (a_1, a_2, a_3) = (b_1 - 1, b_2 - 1, b_3 + 2) \\ \text{or } & (a_1, a_2, a_3) = (b_1 - 1, b_2, b_3 + 1) \end{aligned}$$

Each of these six cases implies that $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ i.e. $a = b$, as wanted to prove point 1. of the proposition. The two other points can be established exactly in the same fashion. \square

Example 2. Using the formula for $\alpha(gh, g_1, g_2)$ given at the end of example 1, we see that if $(g_1, g_2) = (X, Y)$, then the only non trivial value $\alpha(gh, g_1, g_2)$ is given by

$$\alpha(XY, X, Y) = \frac{1}{12}(2f(H, X, Y) + f(X, H, Y) - f(Y, X, H)) = \frac{1}{6}f(X, Y, H) = \frac{4}{3}$$

5 Application

Here we apply the construction of the abelian representative of the equivalence class of the crossed module associated with a 3-cohomology class with values in the trivial module to the construction of quasi-invariant tensors, see [2].

5.1 Preliminaries

Let $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$ be a crossed module. The diagonal map $U\mathfrak{n} \rightarrow U(\mathfrak{n}^{\oplus n})$, $x \mapsto x \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes x$ will be denoted by Δ^n . Denote by \mathfrak{e} the semi-direct product Lie algebra $\mathfrak{m} \rtimes \mathfrak{n}$.

The Lie algebra morphism $\beta : \mathfrak{e} \rightarrow \mathfrak{n}$ given for $X \in \mathfrak{n}$ and $v \in \mathfrak{m}$ by

$$\beta(v, X) = \mu(v) + X,$$

extends to an algebra morphism $\beta : U\mathfrak{e} \rightarrow U\mathfrak{n}$ and further to $\beta : U(\mathfrak{e}^{\oplus n}) \rightarrow U(\mathfrak{n}^{\oplus n})$.

Let A_n be the smallest vector subspace of $U(\mathfrak{e}^{\oplus n})$ containing all elements of the form xvy with $x, y \in U(\mathfrak{n}^{\oplus n})$ and $v \in \mathfrak{m}^{\oplus n}$.

Definition 4. The vector space $\mathcal{U}^{(n)}$ is the quotient of A_n with respect to the relations

$$x\mu(u)yvz = xuy\mu(v)z$$

for all $x, y, z \in U(\mathfrak{n}^{\oplus n})$ and all $u, v \in \mathfrak{m}^{\oplus n}$.

We suggest that the reader reads some examples in Section 4.1 of [2] in order to better understand this notion. The upshot of the vector space $\mathcal{U}^{(n)}$ is the following Lemma:

Lemma 4 (Lemma 27 in [2]). *For all $a, b \in \mathcal{U}^{(n)}$, we have*

$$\beta(a)b = a\beta(b).$$

Finally, let us introduce an action of \mathfrak{m} on $\mathfrak{n} \otimes \mathfrak{n}$: For all $a \in \mathfrak{m}$ and all $r = \sum_i s_i \otimes t_i \in \mathfrak{n} \otimes \mathfrak{n}$, we set

$$a \cdot r = - \sum_i (s_i \cdot a) \otimes t_i + s_i \otimes (t_i \cdot a),$$

where $s_i \cdot a$ is the action of \mathfrak{n} on \mathfrak{m} stemming from the crossed module $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$. This definition is consistent with seeing all elements as elements of the semi-direct product and acting by the bracket.

Now we are ready for the main definition:

Definition 5. *A quasi-invariant tensor in $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$ is a triple (r, c, ξ) where:*

- (a) $r = \sum_q s_q \otimes t_q$ is a symmetric tensor (i.e. $\sum_q s_q \otimes t_q = \sum_q t_q \otimes s_q$);
- (b) $\xi : \mathfrak{n} \rightarrow (\mathfrak{n} \otimes \mathfrak{m}) \oplus (\mathfrak{m} \otimes \mathfrak{n})$ is a linear map, whose image is symmetric, i.e.

$$\xi(X) = \sum_a \xi_a(X) \mu'_a \otimes \mu''_a + \sum_a \xi_a(X) \mu''_a \otimes \mu'_a$$

with $\mu'_a \in \mathfrak{n}$ and $\mu''_a \in \mathfrak{m}$.

- (c) c is an \mathfrak{n} -invariant element in $\ker(\mu) \subset \mathfrak{m}$.

These data is supposed to satisfy the following conditions:

- (a) $X \cdot r = \beta(\xi(X))$ for all $X \in \mathfrak{n}$,
- (b) $u \cdot r = \xi(\mu(u))$ for all $u \in \mathfrak{m}$,
- (c) $\xi([X, Y]) = X \cdot \xi(Y) - Y \cdot \xi(X)$ for all $X, Y \in \mathfrak{n}$.

Cirio and Martins show in [2], Theorem 31, how to construct from a quasi-invariant tensor a totally symmetric infinitesimal 2-braiding in some strict linear 2-category \mathcal{C}_μ associated with the crossed module $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$. This is seen as a categorification of the infinitesimal braiding in the category of \mathfrak{n} -modules which is associated to an r-matrix for \mathfrak{n} , i.e. a symmetric \mathfrak{n} -invariant tensor $r \in \mathfrak{n} \otimes \mathfrak{n}$.

Cirio and Martins go on constructing a quasi-invariant tensor for a certain crossed module corresponding to the Cartan cocycle of $\mathfrak{sl}_2(\mathbb{C})$. Let us show here how one may construct in general a quasi-invariant tensor using the abelian representatives of the equivalence class of the crossed module $\mu : \mathfrak{m} \rightarrow \mathfrak{n}$ which we have constructed earlier.

5.2 Construction of quasi-invariant tensors

Suppose given a crossed module of the form $\mu : V_2 \rightarrow V_3 \times_\alpha \mathfrak{g}$, i.e. it is spliced together from a short exact sequence of \mathfrak{g} -modules

$$0 \rightarrow V = V_1 \xrightarrow{i} V_2 \xrightarrow{d} V_3 \rightarrow 0, \quad (14)$$

and an abelian extension $V_3 \times_\alpha \mathfrak{g}$ of \mathfrak{g} by V_3 via the cocycle $\alpha \in Z^2(\mathfrak{g}, V_3)$, see Theorem 3. We will always suppose that $V = V_1 = \mathbb{C}$ is the trivial 1-dimensional \mathfrak{g} -module. Observe that this means that there is an element $1_{V_2} \in V_2$ which generates the trivial \mathfrak{g} -submodule $i(\mathbb{C})$ and the kernel of d . We denote by $Q : V_3 \rightarrow V_2$ a linear section of the quotient map d . We write $\overline{X} = (0, X) \in V_3 \times_\alpha \mathfrak{g}$ for all $X \in \mathfrak{g}$ and $\overline{h} = (h, 0) \in V_3 \times_\alpha \mathfrak{g}$ for all $h \in V_3$. With these notations, we have for example $\mu(v) = d(v)$ for all $v \in V_2$.

Introduce further the cochains showing up when computing the image of α under the connecting homomorphism associated to the short exact sequence (14). There is the 2-cochain $\omega : \Lambda^2 \mathfrak{g} \rightarrow V_2$ given for all $X, Y \in \mathfrak{g}$ by $Q(\alpha(X, Y)) = w(X, Y)$, and further $\Phi : \Lambda^3 \mathfrak{g} \rightarrow V_2$ given by $\Phi = d_{CE}\omega$. By definition of the connecting homomorphism (and Theorem 3), the 3-class $[\gamma] = \partial[\alpha] \in H^3(\mathfrak{g}; V)$ is represented by γ such that for all $X, Y, Z \in \mathfrak{g}$, $i(\gamma(X, Y, Z)) = \Phi(X, Y, Z)$. All this can be visualized in the following diagram:

$$\begin{array}{ccccc}
 & & \omega = Q\alpha & \xrightarrow{d} & \alpha \\
 & & \downarrow & & \downarrow \\
 C^2(\mathfrak{g}, V_1) & \xrightarrow{i} & C^2(\mathfrak{g}, V_2) & \xrightarrow{d} & C^2(\mathfrak{g}, V_3) \\
 \downarrow d_{CE} & & \downarrow d_{CE} & & \downarrow d_{CE} \\
 C^3(\mathfrak{g}, V_1) & \xrightarrow{i} & C^3(\mathfrak{g}, V_2) & \xrightarrow{d} & C^3(\mathfrak{g}, V_3) \\
 & & \downarrow & & \downarrow \\
 \partial\alpha = \gamma & \xrightarrow{i} & d_{CE}\omega = \Phi & \xrightarrow{d} & 0 = d_{CE}\alpha
 \end{array}$$

The computation in Equation (80) in [2] goes through in this more general framework to show that in $\mathcal{U}^{(2)}$:

$$1_{V_2} \otimes [\overline{X}, \overline{Y}] = 1_{V_2} \otimes \overline{[X, Y]}.$$

Consider now an r-matrix for \mathfrak{g} , i.e. a \mathfrak{g} -invariant, symmetric tensor $r \in \mathfrak{g} \otimes \mathfrak{g}$. We write its components as

$$r = \sum_i s_i \otimes t_i.$$

Let us define the following *compatibility condition*: We have for all $X, Y \in \mathfrak{g}$

$$\sum_i \Phi(s_i, X, Y) \otimes t_i = 1_{V_2} \otimes [X, Y], \text{ and } \sum_i s_i \otimes \Phi(t_i, X, Y) = [X, Y] \otimes 1_{V_2}. \quad (15)$$

Now in order to construct a quasi-invariant tensor, we consider the following objects:

- (a) $\bar{r} = \sum_i \bar{s}_i \otimes \bar{t}_i$ the lift of r to $V_3 \times_\alpha \mathfrak{g} =: E$;
- (b) the map $\xi : E \rightarrow (E \otimes V_2) \oplus (V_2 \otimes E)$ defined as $\xi = -\xi_0 - C$ where
 - (α) $\xi_0(\bar{X}) = \sum_i \omega(s_i, X) \otimes \bar{t}_i + \bar{s}_i \otimes \omega(t_i, X)$,
 - (β) $\xi_0(\bar{h}) = \sum_i (s_i \cdot Q(h)) \otimes \bar{t}_i + \bar{s}_i \otimes (t_i \cdot Q(h))$, and
 - (γ) $C(h, X) = 1_{V_2} \otimes \bar{X} + \bar{X} \otimes 1_{V_2}$.
- (c) $c \in \ker(\mu) = \mathbb{C} \subset V_2$ (via the embedding i).

Theorem 4. *Suppose that in the above situation the compatibility condition (15) is satisfied.*

Then the triple (\bar{r}, ξ, c) is a quasi-invariant tensor for the crossed module $\mu : V_2 \rightarrow V_3 \times_\alpha \mathfrak{g}$.

The proof of the theorem is a simply adaptation of the proof of Theorem 35 in [2]:

Proof. We need to check conditions (a)-(c) in Definition 5. We start with condition (a):

$$\begin{aligned}
 \bar{X} \cdot \bar{r} &= \sum_i ((\alpha(X, s_i), [X, s_i])) \otimes \bar{t}_i + \sum_i \bar{s}_i \otimes ((\alpha(X, t_i), [X, t_i])) \\
 &= \sum_i \overline{\alpha(X, s_i)} \otimes \bar{t}_i + \sum_i \bar{s}_i \otimes \overline{\alpha(X, t_i)} \\
 &= -\beta(\xi_0(\bar{X})) = \beta(\xi(\bar{X})),
 \end{aligned}$$

where we have used in this order the definition of the bracket in the abelian extension, the invariance of r , the relation of α to ω (more precisely,

$$\beta(\omega(s_i, X) \otimes \bar{t}_i) = \mu(\omega(s_i, X)) \otimes \bar{t}_i = -\overline{\alpha(X, s_i)} \otimes \bar{t}_i \quad),$$

and finally that $\mu(1_{V_2}) = 0$.

$$\begin{aligned}
 \bar{h} \cdot \bar{r} &= -\sum_i (\overline{s_i \cdot h} \otimes \bar{t}_i + \bar{s}_i \otimes \overline{t_i \cdot h}) \\
 &= -\sum_i \beta((s_i \cdot Q(h)) \otimes \bar{t}_i + \bar{s}_i \otimes (t_i \cdot Q(h))) = -\beta(\xi_0(\bar{h})) = \beta(\xi(\bar{h})),
 \end{aligned}$$

where we have used in this order the fact that the bracket in the semi-direct product is here just given by the action, then using axiom (a) of a crossed module

$$\beta((s_i \cdot Q(h)) \otimes \bar{t}_i) = \mu(s_i \cdot Q(h)) \otimes \bar{t}_i = [s_i, h] \otimes \bar{t}_i = (s_i \cdot h) \otimes \bar{t}_i,$$

and finally the definitions of ξ_0 and ξ .

Now let us verify condition (b): For all $a \in V_2$, we have

$$a \cdot \bar{r} - \xi(\mu(a)) = \sum_i (s_i \cdot (Q(d(a)) - a)) \otimes \bar{t}_i + \bar{s}_i \otimes (t_i \cdot (Q(d(a)) - a)),$$

and this vanishes since $Q(d(a)) - a$ is in $\ker(d) = \mathbb{C}$ which is \mathfrak{g} -invariant.

Finally, we verify condition (c): For this, we start by computing:

$$\begin{aligned} \xi_0([\bar{X}, \bar{Y}]) &= \xi_0(\overline{[X, Y]}) - \xi_0(\overline{\alpha(X, Y)}) \\ &= \sum_i (\omega(s_i, [X, Y]) \otimes \bar{t}_i + \bar{s}_i \otimes \omega(t_i, [X, Y]) + \\ &\quad + (s_i \cdot Q\alpha(X, Y)) \otimes \bar{t}_i + \bar{s}_i \otimes (t_i \cdot Q\alpha(X, Y))) \end{aligned}$$

$$\bar{X} \cdot \xi_0(\bar{Y}) = \sum_i (X \cdot \omega(s_i, Y)) \otimes \bar{t}_i + \omega(s_i, Y) \otimes [\bar{X}, \bar{t}_i] + [\bar{X}, \bar{s}_i] \otimes \omega(t_i, Y) + \bar{s}_i \otimes (X \cdot \omega(t_i, Y))$$

$$\bar{Y} \cdot \xi_0(\bar{X}) = \sum_i (Y \cdot \omega(s_i, X)) \otimes \bar{t}_i + \omega(s_i, X) \otimes [\bar{Y}, \bar{t}_i] + [\bar{Y}, \bar{s}_i] \otimes \omega(t_i, X) + \bar{s}_i \otimes (Y \cdot \omega(t_i, X))$$

We now claim that

$$\begin{aligned} \xi_0([\bar{X}, \bar{Y}]) - \bar{X} \cdot \xi_0(\bar{Y}) + \bar{Y} \cdot \xi_0(\bar{X}) &= \sum_i \Phi(s_i, X, Y) \otimes \bar{t}_i + \bar{s}_i \otimes \Phi(t_i, X, Y) \\ &= \overline{[X, Y]} \otimes 1_{V_2} + 1_{V_2} \otimes \overline{[X, Y]}. \end{aligned}$$

where we used the compatibility condition (15) to rewrite the last expression.

Reminding that $\Phi = d_{CE}\omega$, we obtain

$$\begin{aligned} &\xi_0([\bar{X}, \bar{Y}]) - \bar{X} \cdot \xi_0(\bar{Y}) + \bar{Y} \cdot \xi_0(\bar{X}) - \sum_i \Phi(s_i, X, Y) \otimes \bar{t}_i + \bar{s}_i \otimes \Phi(t_i, X, Y) = \\ &= \sum_i -\omega(s_i, Y) \otimes [\bar{X}, \bar{t}_i] - [\bar{X}, \bar{s}_i] \otimes \omega(t_i, Y) + \omega(s_i, X) \otimes [\bar{Y}, \bar{t}_i] + [\bar{Y}, \bar{s}_i] \otimes \omega(t_i, X) + \\ &\quad -\omega(X, [Y, s_i]) \otimes \bar{t}_i - \omega(Y, [s_i, X]) \otimes \bar{t}_i - \bar{s}_i \otimes \omega(X, [Y, t_i]) - \bar{s}_i \otimes \omega(Y, [t_i, X]). \end{aligned}$$

We now expand each term of the form $[\bar{X}, \bar{t}_i]$ into $\overline{\alpha(X, t_i)} + \overline{[X, t_i]}$. The terms involving the cocycle α are:

$$\sum_i -\omega(s_i, Y) \otimes \overline{\alpha(X, t_i)} - \overline{\alpha(X, s_i)} \otimes \omega(t_i, Y) + \omega(s_i, X) \otimes \overline{\alpha(Y, t_i)} + \overline{\alpha(Y, s_i)} \otimes \omega(t_i, X)$$

These terms cancel pairwise using Lemma 4 and the definition of ω . For example:

$$\omega(s_i, Y) \otimes \overline{\alpha(X, t_i)} = \omega(s_i, Y) \otimes \mu(Q(\alpha(X, t_i))) = \mu(\omega(s_i, Y)) \otimes \omega(X, t_i) = \overline{\alpha(s_i, Y)} \otimes \omega(X, t_i).$$

The remaining eight terms not containing α cancel pairwise thanks to the invariance property of r . Namely, applying the linear map $\omega(Y, -) \otimes \text{id}$ to the invariance relation

$$s_i \otimes \overline{[X, t_i]} + [X, s_i] \otimes \bar{t}_i,$$

we obtain (using the anti-symmetry of ω)

$$-\omega(s_i, Y) \otimes \overline{[X, t_i]} - \omega(Y, [s_i, X]) \otimes \bar{t}_i.$$

This proves the claim. We go on with:

$$\begin{aligned} C([\overline{X}, \overline{Y}]) - \overline{X} \cdot C(\overline{Y}) + Y \cdot C(\overline{X}) &= C(\alpha(X, Y), \overline{[X, Y]}) - \overline{X} \cdot C(\overline{Y}) + Y \cdot C(\overline{X}) = \\ &= -1_{V_2} \otimes \overline{[X, Y]} - \overline{[X, Y]} \otimes 1_{V_2}. \end{aligned}$$

Thus summing over all contributions proves condition (c) for $\xi([\overline{X}, \overline{Y}])$.

Now we show condition (c) for $\xi([\overline{X}, \overline{h}])$. We start by computing:

$$\xi_0([\overline{X}, \overline{h}]) = \xi_0(\overline{X \cdot h}) = \sum_i (s_i \cdot Q(\overline{X \cdot h})) \otimes \bar{t}_i + \bar{s}_i \otimes (t_i \cdot Q(\overline{X \cdot h})),$$

$$\overline{X} \cdot \xi_0(\overline{h}) = \sum_i (\overline{X \cdot (s_i \cdot Q(h))}) \otimes \bar{t}_i + (s_i \cdot Q(h)) \otimes [\overline{X}, \bar{t}_i] + [\overline{X}, \bar{s}_i] \otimes (t_i \cdot Q(h)) + \bar{s}_i \otimes (X \cdot (t_i \cdot Q(h))),$$

$$\overline{h} \cdot \xi_0(\overline{X}) = \sum_i -\omega(s_i, X) \otimes (\overline{t_i \cdot h}) - (\overline{s_i \cdot h}) \otimes \omega(t_i, X).$$

Observe that $X \cdot (Y \cdot Q(h)) = X \cdot (Q(Y \cdot h))$; i.e. Q is \mathfrak{g} -equivariant when subject to a \mathfrak{g} -action. This follows from

$$\mu(Y \cdot Q(h) - Q(Y \cdot h)) = Y \cdot (\mu(Q(h))) - Y \cdot h = 0,$$

because $\ker[\mu]$ is \mathfrak{g} -invariant.

All this is used to write

$$\begin{aligned} &\xi_0([\overline{X}, \overline{h}]) - \overline{X} \cdot \xi_0(\overline{h}) + \overline{h} \cdot \xi_0(\overline{X}) = \\ &= \sum_i -([X, s_i] \cdot Q(h)) \otimes \bar{t}_i - \bar{s}_i \otimes ([X, t_i] \cdot Q(h)) - (s_i \cdot Q(h)) \otimes [\overline{X}, \bar{t}_i] - \\ &\quad [\overline{X}, \bar{s}_i] \otimes (t_i \cdot Q(h)) - \omega(s_i, X) \otimes (\overline{t_i \cdot h}) - (\overline{s_i \cdot h}) \otimes \omega(t_i, X) \\ &= \sum_i (-([X, s_i] \cdot Q(h)) \otimes \bar{t}_i + (s_i \cdot Q(h)) \otimes [\overline{X}, \bar{t}_i] + \\ &\quad -(\bar{s}_i \otimes ([X, t_i] \cdot Q(h)) + [\overline{X}, \bar{s}_i] \otimes (t_i \cdot Q(h))) + \\ &\quad - (s_i \cdot Q(h)) \otimes \overline{\alpha(X, t_i)} - \overline{\alpha(X, s_i)} \otimes (t_i \cdot Q(h)) - \omega(s_i, X) \otimes \overline{t_i \cdot h} - \overline{s_i \cdot h} \otimes \omega(t_i, X). \end{aligned}$$

In the two big parentheses, we use as before the invariance of r , and therefore they vanish. The four terms in the last line cancel pairwise, using once again

Lemma 4. This completes the proof. \square

At last, let us address the question for which crossed modules $\mu : V_2 \rightarrow V_3 \times_{\alpha} \mathfrak{g}$ and which r -matrices r , the compatibility condition is fulfilled. Observe that one example (i.e. $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$) is certainly the one given in [2]. This holds more generally for finite dimensional metric Lie algebras, thanks to the following

Proposition 6. *Let (\mathfrak{g}, κ) be a finite dimensional metric Lie algebra of dimension n , $x_* := (x_i)_{1 \leq i \leq n}$ be a basis of \mathfrak{g} and denote by $x^* := (x^i)_{1 \leq i \leq n}$ the κ -orthonormal basis to x_* , which is uniquely defined by requiring that*

$$\kappa(x_i, x^j) = \delta_{i,j} \quad , \quad 1 \leq i, j \leq n,$$

where $\delta_{i,j}$ denotes the Kronecker symbol. Then, the 2-tensor $r(x_*)$ defined by

$$r(x_*) := \frac{1}{2} \sum_{i=1}^n (x_i \otimes x^i + x^i \otimes x_i) \in \mathfrak{g} \otimes \mathfrak{g}$$

is an r -matrix for \mathfrak{g} which satisfies the compatibility condition (15) of Cirio-Martins with respect to the Cartan cocycle $\kappa([-,-], -)$. Moreover, $r(x_*)$ does not depend on the choice of the basis x_* and will be written r_{κ} in the sequel.

Proof. Let us first prove that $r(x_*)$ is an r -matrix. It is obviously symmetric by definition, so it remains to show that it is invariant. This is the content of Proposition XVII.1.3 of [5] in the case when κ is the Killing form of a semi-simple Lie algebra, and the proof given there, even if it still works for an arbitrary metric Lie algebra, relies on the fact that the Casimir element metric Lie algebra is central in its universal enveloping algebra, which we don't need here (even if it also remains true for an arbitrary metric Lie algebra). Thus we propose here an alternative direct proof: First, observe that for any z in \mathfrak{g} , the fact that x_* is κ -orthonormal to x^* implies that

$$z = \sum_{i=1}^n \kappa(z, x^i) x_i \tag{16}$$

Moreover, since κ is symmetric, we also have that

$$z = \sum_{i=1}^n \kappa(z, x_i) x^i \tag{17}$$

Now let g be an element of \mathfrak{g} . By definition of the diagonal action of \mathfrak{g} on $\mathfrak{g} \otimes \mathfrak{g}$:

$$g \cdot \left(\sum_{i=1}^n x_i \otimes x^i \right) := \sum_{i=1}^n ([g, x_i] \otimes x^i + x_i \otimes [g, x^i])$$

Applying equations (16) with $z := [g, x_i]$ and (17) with $z = [g, x^i]$ respectively to the first summand involving $[g, x_i]$'s and the second summand involving $[g, x^i]$'s,

we can rewrite the preceeding equation as

$$g \cdot \left(\sum_{i=1}^n x_i \otimes x^i \right) = \sum_{i=1}^n \sum_{j=1}^n (\kappa([g, x_i], x^j) x_j \otimes x^i + x_i \otimes \kappa([g, x^i], x_j) x^j)$$

Using the invariance of κ in the first summand and the fact that it is symmetric after having permuted i with j in the second, we get that

$$g \cdot \left(\sum_{i=1}^n x_i \otimes x^i \right) = \sum_{i=1}^n \sum_{j=1}^n -(\kappa(x_i, [g, x^j]) x_j \otimes x^i + x_j \otimes \kappa(x_i, [g, x^j]) x^i) = 0$$

since values of κ can be moved from one side of the tensor product to the other as the tensor product is taken over \mathbb{C} . The same arguments apply to prove that

$$g \cdot \left(\sum_{i=1}^n x^i \otimes x_i \right) = 0$$

Thus, $r(x_*)$ appears as the sum of two invariant terms i.e.

$$g \cdot r(x_*) = g \cdot \left(\frac{1}{2} \sum_{i=1}^n x_i \otimes x^i + x^i \otimes x_i \right) = 0 + 0 = 0$$

for all g in \mathfrak{g} , which shows that $r(x_*)$ is indeed \mathfrak{g} -invariant.

To establish that $r(x_*)$ fulfills the compatibility condition (15) with respect to the Cartan cocycle $\kappa([- , -], -)$, observe that for any crossed module

$$0 \rightarrow \mathbb{C} \xrightarrow{i} \mathfrak{m} \rightarrow \mathfrak{n} \rightarrow \mathfrak{g} \rightarrow 0$$

representing the Cartan cocycle, the associated cochain $\Phi : \Lambda^3 \mathfrak{g} \rightarrow \mathfrak{m}$ satisfies $\Phi(X, Y, Z) = i(\gamma(X, Y, Z))$. So, up to an identification of \mathbb{C} with the subspace of \mathfrak{m} generated by $1_{\mathfrak{m}} := i(1_{\mathbb{C}})$, Φ is equal to the Cartan cocycle. Therefore, applying equation (16) to $z := [X, Y]$ and the invariance and symmetry of κ , we get

$$1_{\mathfrak{m}} \otimes [X, Y] = \sum_{i=1}^n 1_{\mathfrak{m}} \otimes \kappa([X, Y], x_i) x^i = \sum_{i=1}^n \kappa([x_i, X], Y) \otimes x^i = \sum_{i=1}^n \Phi(x_i, X, Y) \otimes x^i.$$

Similarly, using equation (17) instead of equation (16) leads to

$$1_{\mathfrak{m}} \otimes [X, Y] = \sum_{i=1}^n x_i \otimes \Phi(x^i, X, Y),$$

which, averaged with the previous equation, gives

$$1_{\mathfrak{m}} \otimes [X, Y] = \frac{1}{2} \sum_{i=1}^n (x_i \otimes \Phi(x^i, X, Y) + x^i \otimes \Phi(x_i, X, Y)),$$

showing that $r(x_*) = \frac{1}{2} \sum_{i=1}^n (x_i \otimes x^i + x^i \otimes x_i)$ satisfies the left equation of the compatibility condition (15). The right equation can be obtained exactly in the same fashion.

The fact that $r_\kappa := r(x_*)$ does not depend on the choice of x_* is standard in the literature. The proof we propose here in the sake of self-containendness is a straightforward adaptation to the arbitrary metric Lie algebra case of the one of Proposition XVII.1.2 of [5], in which it is shown that, a posteriori, the Casimir element of a semi-simple Lie algebra does not depend on the choice of the basis used to define it.

If $y_* := (y_i)_{1 \leq i \leq n}$ is another basis of \mathfrak{g} with orthonormal basis $y^* := (y^i)_{1 \leq i \leq n}$ we have, using equations (16) and (17) to express the x_i 's and the x^i 's respectively in the basis y_* and y^* :

$$\begin{aligned} \sum_{i=1}^n x_i \otimes x^i &= \sum_{i=1}^n \sum_{1 \leq j, k \leq n} \kappa(x_i, y^j) y_j \otimes \kappa(x^i, y_k) y^k \\ &= \sum_{1 \leq j, k \leq n} \left(\sum_{i=1}^n \kappa(x_i, y^j) \kappa(x^i, y_k) \right) y_j \otimes y^k \end{aligned}$$

But for all j and k in $\{1, \dots, n\}$, equation (17) applied to $z = y^j$ gives

$$\sum_{i=1}^n \kappa(x_i, y^j) \kappa(x^i, y_k) = \kappa \left(\sum_{i=1}^n \kappa(x_i, y^j) x^i, y_k \right) = \kappa(y^j, y_k) = \delta_{jk}$$

which shows that the previous equality can be rewritten as

$$\sum_{i=1}^n x_i \otimes x^i = \sum_{1 \leq j, k \leq n} \delta_{j,k} y_j \otimes y^k = \sum_{j=1}^n y_j \otimes y^j$$

Symmetrizing both sides, we get

$$r(x_*) = r(y_*)$$

so that the standard r -matrix of a metric algebra indeed does not depend on the choice of the basis used to define it. \square

Corollary 2. *Let \mathfrak{g} be a semi-simple complex Lie algebra with Killing form κ . Then the standard r -matrix of \mathfrak{g} is the unique r -matrix of \mathfrak{g} which both lifts its Casimir element and fulfills the compatibility condition (15) with respect to the Cartan cocycle.*

Proof. By definition, the Casimir element Ω of \mathfrak{g} is the image in $U\mathfrak{g}$ of its standard r -matrix $r_\kappa \in \mathfrak{g} \otimes \mathfrak{g}$ under the multiplication map $U\mathfrak{g}^{\otimes 2} \rightarrow U\mathfrak{g}$ restricted to $\mathfrak{g}^{\otimes 2}$, i.e.

$$\Omega = \frac{1}{2} \sum_{i=1}^n (x_i x^i + x^i x_i) \in U\mathfrak{g}$$

for any basis $(x_i)_{1 \leq i \leq n}$ with κ -orthonormal basis $(y_i)_{1 \leq i \leq n}$. Thus, r_κ is certainly a symmetric lift of Ω to $\mathfrak{g}^{\otimes 2}$ and Proposition 6 shows that it is invariant and fulfills the compatibility condition (15) with respect to the Cartan cocycle. Since the associative algebra $U\mathfrak{g}$ is the quotient of the free associative algebra $T\mathfrak{g}$ by a non homogenous ideal, the restriction of its multiplication to $\mathfrak{g}^{\otimes 2}$ is injective, which proves the unicity part of the statement. \square

Corollary 3. *Let \mathfrak{g} be a semi-simple complex Lie algebra and let the crossed module $\mu : V_2 \rightarrow V_3 \times_\alpha \mathfrak{g}$ be any abelian representative for the (cohomology class represented by) Cartan cocycle $\kappa([-,-], \cdot)$. Then the triple (\bar{r}_κ, ξ, c) defined by dots (a), (b) and (c) of section 5.2 is a quasi-invariant tensor for μ .*

Remark 5. (a) *Observe that the short exact sequence considered earlier*

$$0 \rightarrow \mathbb{C} \rightarrow (U\mathfrak{g})^\vee \rightarrow (U\mathfrak{g}^+)^\vee \rightarrow 0$$

gives rise to an abelian representative of the crossed module corresponding to the Cartan cocycle of the form

$$0 \rightarrow \mathbb{C} \rightarrow (U\mathfrak{g})^\vee \rightarrow (U\mathfrak{g}^+)^\vee \times_\alpha \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow 0$$

where α is the lift defined by formula (9). Recall from section 5.2 that the ξ_0 part of the quasi-invariant tensor associated to the standard r -matrix $r_\kappa = \sum_i s_i \otimes t_i$ of \mathfrak{g} satisfies

$$\xi_0(g) = \sum_i (\alpha(-, s_i, g) \otimes t_i + s_i \otimes \alpha(-, t_i, g)) \in (U\mathfrak{g})^\vee \otimes \mathfrak{g} \oplus \mathfrak{g} \otimes (U\mathfrak{g})^\vee$$

for all g in $\mathfrak{g} \subset (U\mathfrak{g}^+)^\vee \oplus_\alpha \mathfrak{g}$. Thus $\xi_0(g)$ can be thought of as the (direct) sum $\xi_0^1(g) \oplus \xi_0^2(g)$ of two elements of $\text{Hom}_{\mathbb{C}}(U\mathfrak{g}, \mathfrak{g})$. Choosing $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ with the standard basis $\{X, Y, H\}$ defined above and $g := Y$, the computation of $\alpha(XY, -, -)$ given in Example 2 shows that

$$\xi_0^1(Y)(XY) = \sum_i \alpha(XY, s_i, Y)t_i = \frac{1}{4}\alpha(XY, X, Y)Y = \frac{1}{3}Y$$

and similarly

$$\xi_0^2(Y)(XY) = \frac{1}{3}Y$$

where we have used that the standard r -matrix of $\mathfrak{sl}_2(\mathbb{C})$ is given by $r = \frac{1}{4}(X \otimes Y + Y \otimes X + \frac{1}{2}H \otimes H)$.

(b) *Observe that the compatibility relation is also trivially satisfied for all abelian Lie algebras. Moreover, for abelian Lie algebras, the formula (9) can be evaluated explicitly.*

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